Markov Processes with Darning and their Approximations

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Outline

◊Background of the problem;

◊ Definition of Markov processes with darning;

◊ Main Results: Construction of Markov processes with darning;

◊ Further work

Markov boundary theory

Given the minimal Markov process X^0 on domain D, one constructs and characterizes all of Markovian extensions of X^0 such that they spend zero Lebesgue amount of time outside of D.

Brief history on boundary theory

◊ Boundary theory for one-dimensional diffusions is well understood thanks to the fundamental work of Feller (1954). —Analytic

◊ For Markov chain with countable states, significant progresses has been made by K.L.Chung (1970), Z.T.Hou (1978), M.F.Chen (1986) and X.Q.Yang (1990),etc.

◊ K.Ito (1970) use excursion theory and Poisson point processes to construct general Markovian extension of the minimal diffusions.—Probabilistic.

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◊ In Chen and Fukushima (2008), Markovian extensions of the minimal processes are carried out through Poisson point processes of excursions.

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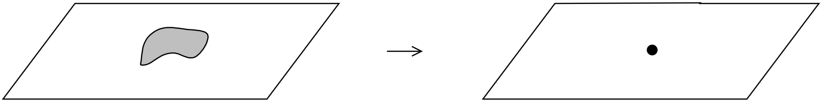
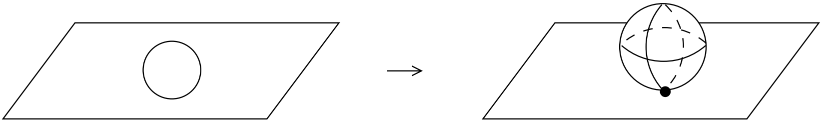
What is Brownian motion with darning?

(i) $N = 1$ and K is a non-polar connected compact subset of $\mathbb{R}^d$.



(ii) $N = 1$ and $K = \partial B(0, 1)$. D^* is homeomorphic to the plane with a sphere sitting on top of it.



What is Brownian motion with darning?

$E \subset \mathbb{R}^d$, $A_1, \dots, A_N$ are disjoint compact subsets of E. Let $D = E \setminus \cup_{j=1}^N A_j$. BMD on $E^* := D \cup \{a_1^*, \dots, a_N^*\}$ is a Brownian motion in E by "shorting" each A_j into a single point a_j^*.

Definition

Brownian motion with darning (BMD) X^* is an m-symmetric diffusion on E^* such that
(i) its part process in D has the same law as W^D;
(ii) it admits no killings on K^*.

It follows that X^* spends zero Lebesgue amount of time at K^*.

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Chen-Fukushima (2007, 2011): general compact A_j.

Theorem (Chen-Fukushima, 2011)

BMD exists and is unique in law.

Two dimensional BMD has conformal invariance and can be used to study Komatu-Loewner equations in multiply connected domains. Chen-Fukushima-Rohde, Chen-Fukushima, Chen-Fukushima-Suzuki.

Construction:

(1) X^* can be constructed by using Poisson point process of excursions of killed Brownian motion in D.

(2) A more direct way is through Dirichlet form method. Chen (2012), Chen-Fukushima-Rohde (TAMS 2016).

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What is Markov process with darning?

Let X be an m-symmetric strong Markov process on E with $\mathrm{supp}[m] = E$. Suppose $A_1, \ldots, A_N$ are disjoint compact subsets of E. Let $D = E \setminus \cup_{j=1}^N A_j$. Intuitively speaking, Markov process with darning X^* on $E^* := D \cup \{a_1^*, \ldots, a_N^*\}$ is a Markov process in E by “shorting" each A_j into a single point a_j^*.

Denote $\{a_1^*, \ldots, a_N^*\}$ by K^*. Let m^* be the measure on E^* defined by $m^*(A) = m(A \cap D)$.

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Definition

Markov process with darning (MPD) X^* is an m^*-symmetric Markov process on E^* such that
(i) its part process of X^* in D has the same law as X^D;
(ii) The jumping measure $J^*(dx, dy)$ and killing measure κ^* of X^* on E^* have the properties inherited from X without incurring additional jumps or killings, that is,

$$J^* = J \text{ on } D \times D, \quad J^*(a_i^*, dy) = J(A_i, dy) \text{ on } D,$$

$$J^*(a_i^*, a_j^*) = J(A_i, A_j),$$

$$\kappa^* = \kappa \quad \text{on } D \quad \text{and} \quad \kappa^*(a_j^*) = \kappa(A_j).$$

It follows that X^* spends zero Lebesgue amount of time at K^*.

Goal

Markov processes with darning have been constructed in Chen-Fukushima-Ying (2006) and Chen-Fukushima (2012) using excursion theory, under an assumption that the original Markov process enters these compact subsets (holes) in a continuous way.

The goal of this talk (and the paper) is to

(i) remove this restriction by using a new approach;

(ii) develop approximation schemes for general Markov processes with darning by more concrete processes, which can be used in simulation.

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Approximation by introducing jumps

Intuitively, when the jumping intensity for these additional jumps increases to infinity, the new process can no longer distinguish points among each A_j, which would result in shorting (or darning) each A_j into a single point a_j^*.

For each $\lambda > 0$, consider

$$\mathcal{E}^{(\lambda)}(u,u) = \mathcal{E}(u,u) + \lambda \sum_{j=1}^{N} \int_{A_j \times A_j} (u(x) - u(y))^2 \mu_j(dx)\mu_j(dy)$$

for $u \in \mathcal{F}$. It is easy to see that $(\mathcal{E}^{(\lambda)}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E; m)$ and it associates an m-symmetric Hunt process $X^{(\lambda)}$. The process $X^{(\lambda)}$ is the superposition of X with jumps among points within each A_j.

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Approximation by introducing jumps

$X^{(\lambda)}$ can also be obtained from X by the following piecing together procedure. Let X^0 be the subprocess obtained from X through killing via measure $\lambda \sum_{j=1}^{N} \mu_j$.

- Run a copy of X^0 starting from x and set $X_t^{(\lambda)} = X_t^0$ for $t \in [0, T_1)$, where $T_1 = \zeta^0$ is the lifetime of X^0 starting from x.
- If $\zeta^0 = \infty$ or $X_{T_1-}^{(\lambda)} = \partial$, then we define $X_t^{(\lambda)} = \partial$ for $t \geq T_1$. Otherwise, $X_{T_1-}^{(\lambda)} \in F := \cup_{j=1}^{N} A_j$, say $X_{T_1-}^{(\lambda)} \in A_{j_1}$. Select $x_1 \in A_{j_1}$ according to the probability distribution $\mu_{j_1}/\mu_{j_1}(A_j)$ and define $X_{T_1}^{(\lambda)} = x_1$.
- Run an independent copy of X^0 starting from x_1 ...
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Limiting process

◊ When $\lambda \to \infty$, process $X^{(\lambda)}$ behaves like X outside F but can not distinguish points in each A_j. In other words, in the limit, each A_j is collapsed into a single point a_j^*.

◊ If the limit exists, the limiting process should be Markov process with darning of X but up to a time change.

This is because m is a symmetrizing measure for each $X^{(\lambda)}$ so under stationarity, each $X^{(\lambda)}$ spends time in F at a rate proportional to $m(F)$. So the limiting process is a sticky MPD on E^* obtained from X^* through a time change via Revuz measure

$$\mu = m|_D + \sum_{j=1}^{N} m(A_j)\delta_{\{a_j^*\}}.$$

Difficulty: Topological problem. Sudden collapse of holes at $\lambda = \infty$.

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Closed symmetric form

Let $(\mathcal{E}, \mathcal{F})$ be a closed symmetric form on on $L^2(E; m)$; that is, $\mathcal{F}$ is a linear subspace of $L^2(E; m)$, $\mathcal{E}$ is a non-negative definite symmetric form defined on $\mathcal{F} \times \mathcal{F}$ such that $\mathcal{F}$ is a Hilbert space with inner product $\mathcal{E}_1$.

Resolvent on $L^2(E; m)$: for every $f \in L^2(E; m)$ and $\alpha > 0$, there is a unique $G_\alpha f \in \mathcal{F}$ such that

$$\mathcal{E}_\alpha(G_\alpha f, g) = (f, g)_{L^2(E;m)} \quad \text{for every } g \in \mathcal{F}.$$

Fact: the resolvent $\{G_\alpha, \alpha > 0\}$ of $(\mathcal{E}, \mathcal{F})$ is strongly continuous (that is, $\lim_{\alpha\to\infty} \|\alpha G_\alpha f - f\|_{L^2(E;m)} = 0$ for every $f \in L^2(E; m)$) if and only if $\mathcal{F}$ is dense in $L^2(E; m)$.

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Resolvent

If $\mathcal{F}$ is not dense in $L^2(E; m)$, denote by $\overline{\mathcal{F}}$ the closure of $\mathcal{F}$ in $L^2(E; m)$. Then $(\mathcal{E}, \mathcal{F})$ is a closed symmetric form on $\overline{\mathcal{F}}$. The following facts are known. There is a unique strongly continuous contraction symmetric resolvent $\{\widehat{G}_\alpha; \alpha > 0\}$ on $\overline{\mathcal{F}}$ associated with it:

$$\mathcal{E}_\alpha(\widehat{G}_\alpha f, g) = (f, g)_{L^2(E;m)} \quad \text{for every } g \in \mathcal{F}.$$

So it associates a strongly continuous semigroup $\{\widehat{P}_t; t \geq 0\}$ on $\overline{\mathcal{F}}$.

Let Π be the orthogonal projection operator from $L^2(E; m)$ onto $\overline{\mathcal{F}}$. Then $G_\alpha f = \widehat{G}_\alpha(\Pi f)$.

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Mosco convergence

Definition

A sequence of closed symmetric forms $\{(\mathcal{E}^n, \mathcal{F}^n)\}$ on $L^2(E; m)$ is said to be convergent to a closed symmetric form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ in the sense of Mosco if
(a) For every sequence $\{u_n, n \geq 1\}$ in $L^2(E; m)$ that converges weakly to u in $L^2(E; m)$,

$$\liminf_{n\to\infty} \mathcal{E}^n(u_n, u_n) \geq \mathcal{E}(u, u),$$

(b) For every $u \in L^2(E; m)$, there is a sequence $\{u_n, n \geq 1\}$ in $L^2(E; m)$ converging strongly to u in $L^2(E; m)$ such that

$$\limsup_{n\to\infty} \mathcal{E}^n(u_n, u_n) \leq \mathcal{E}(u, u).$$

Mosco convergence

Proposition (Mosco, 1994)

Let $(\mathcal{E}, \mathcal{F})$ and $\{(\mathcal{E}^n, \mathcal{F}^n), n \geq 1)\}$ be a sequence of closed symmetric forms on $L^2(E; m)$. The following are equivalent:

(i) $(\mathcal{E}^n, \mathcal{F}^n)$ *converges to* $(\mathcal{E}, \mathcal{F})$ *in the sense of Mosco;*

(ii) *For every* $\alpha > 0$ *and* $f \in L^2(E; m)$, $G^n_\alpha f$ *converges to* $G_\alpha f$ *in* $L^2(E; m)$ *as* $n \to \infty$;

(iii) *When* $\mathcal{F}^n$ *and* $\mathcal{F}$ *are all dense in* $L^2(E; m)$, *then (i) is equivalent to the following: For every* $t > 0$ *and* $f \in L^2(E; m)$, $P^n_t f$ *converges to* $P_t f$ *in* $L^2(E; m)$ *as* $n \to \infty$.

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Mosco convergence: new results

Theorem (Chen-Peng, 2017)

Let $(\mathcal{E}, \mathcal{F})$ and $\{(\mathcal{E}^n, \mathcal{F}^n), n \geq 1)\}$ be closed symmetric forms on $L^2(E; m)$. Suppose that $\overline{\mathcal{F}}^n \supset \overline{\mathcal{F}}$ for every $n \geq 1$. Let $(\widehat{P}^n_t; t \geq 0\}$ and $(\widehat{P}_t; t \geq 0\}$ be the semigroups on $\overline{\mathcal{F}}^n$ and $\overline{\mathcal{F}}$ associated with $(\mathcal{E}^n, \mathcal{F}^n)$ and $(\mathcal{E}, \mathcal{F})$. Then

(i) *If $(\mathcal{E}^n, \mathcal{F}^n)$ converges to $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco, then for every $t > 0$ and $f \in \overline{\mathcal{F}}$, $\widehat{P}^n_t f$ converges to $\widehat{P}_t f$ in $L^2(E; m)$.*

(ii) *Suppose that the closed subspace $\overline{\mathcal{F}}^n$ converges to $\overline{\mathcal{F}}$ in $L^2(E; m)$ in the sense that $\lim_{n\to\infty} \|\Pi^n f - \Pi f\|_{L^2(E;m)} = 0$ for every $f \in L^2(E; m)$, where Π^n and Π denote the orthogonal projection operators of $L^2(E; m)$ onto $\overline{\mathcal{F}}^n$ and $\overline{\mathcal{F}}$, respectively. If $\widehat{P}^n_t f$ converges to $\widehat{P}_t f$ in $L^2(E; m)$, for every $t > 0$ and $f \in \overline{\mathcal{F}}$, then $(\mathcal{E}^n, \mathcal{F}^n)$ converges to $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco.*

Sticky MPD

X^*: MPD on E^* and $\mu = m^* + \sum_{j=1}^N m(A_j)\delta_{a_j^*}$, where δ_{a^*} is the Dirac measure concentrated at the point a_j^*. The smooth measure μ determines a positive continuous additive functional A^μ of X^*:

$$A_t^\mu = t + \sum_{j=1}^N m(A_j) L_t^{a_j^*},$$

where $L^{a_j^*}$ is the local time of X^* at a_j^* having Revuz measure $\delta_{a_j^*}$. Let $\tau_t := \inf\{s > 0 : A_s^\mu > t\}$ and $Y_t = X_{\tau_t}^*$.

Theorem (Chen-Peng, 2017)

For any increasing sequence $\{\lambda_n, n \geq 1\}$ of positive real numbers that increases to infinity, the Dirichlet form $(\mathcal{E}^{(\lambda_n)}, \mathcal{F})$ is Mosco convergent to the closed symmetric form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E; m)$.

Sticky MPD

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Convergence in finite dimensional sense

We can then deduce that $X^{(\lambda_n)}$ converges to Y in the following finite-dimensional sense.

Theorem (Chen-Peng, 2017)

For every $0 = t_0 \leq t_1 < \cdots t_k < \infty$ *and bounded* $\{f_j; 1 \leq j \leq k\} \subset \overline{\widetilde{\mathcal{F}}}$,

$$\lim_{n\to\infty} \mathbb{E}^n_m \left[\prod_{j=0}^{k} f_j(X^n_{t_j}) \right] = \mathbb{E}^*_\mu \left[\prod_{j=0}^{k} f^*_j(Y_{t_j}) \right],$$

where $\widetilde{\mathcal{F}}$ *is identified with* $\mathcal{F}^*$.

Further work

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◊ Invariance principle for MPD (Discrete and Continuous))

◊ Stability of Markov processes with darning on shorting domains

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Thank you!